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The reduction of the closest disentangled states

Satoshi Ishizaka

Fundamental Research Laboratories, NEC Corporation, 34 Miyukigaoka, Tsukuba, Ibaraki, 305-8501, Japan
and
CREST, Japan Science and Technology Corporation (JST), 3-13-11 Shibuya, Shibuya-ku, Tokyo, 150-0002, Japan

E-mail: isizaka@frl.cl.nec.co.jp

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Abstract
We study the closest disentangled state to a given entangled state in any system (multi-party with any dimension). We obtain the set of equations the closest disentangled state must satisfy, and show that its reduction is strongly related to the extremal condition of the local filtering on each party. Although the equations we obtain are not still tractable, we find some sufficient conditions for which the closest disentangled state has the same reduction as the given entangled state. Further, we suggest a prescription to obtain a tight upper bound of the relative entropy of entanglement in two-qubit systems.

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Quantum entanglement is the most striking feature of quantum mechanics. Intensive challenges to harness the power of entanglement as a physical resource have continued. In order to quantify the resource of entanglement, several measures such as the entanglement of formation [1] (or entanglement cost), entanglement of distillation [1], relative entropy of entanglement [2, 3], have been proposed.

The relative entropy of entanglement is defined as the distance to the disentangled state closest to the given entangled state under the measure of relative entropy. This implies that the closest disentangled state plays an important role in quantifying the quantum entanglement. In addition, the closest disentangled state itself answers the following question: what is the state when the quantum correlation is completely but minimally (maintaining the classical correlation as long as possible [3]) washed out? Therefore, it will be important to clarify the properties of the closest disentangled state itself to understand the characteristics of quantum entanglement.

Further, the analytical formula of the relative entropy of entanglement has been strongly required to clarify the relations between entanglement and the performance of many applications of quantum information [4, 5]. However, deriving the analytical formula has been known to be a hard problem even in the simplest two-qubit system. Mathematically,

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the difficulty lies in searching for the closest disentangled state on the complicated boundary surface of the set of disentangled states in the Hilbert space. Therefore, to investigate the closest disentangled state might also be important in the sense that it might give some hints for solving the difficult problem.

In this paper we consider the physical operation of local filtering in order to investigate the properties of the closest disentangled states. This physical operation ensures that the state after the operation is disentangled if the state before operation is disentangled. As a result, we can obtain some equations the closest disentangled state must satisfy, although the geometry of the entangled–disentangled boundary is quite complicated. In particular, we show that the reduction of the closest disentangled state is strongly related to the extremal condition of the local filtering on each party. Although the equations we obtain are not still tractable, we find some sufficient conditions for which the closest disentangled state has the same reduction as the given entangled state. Further, in the case of two qubits, we suggest a prescription to obtain an upper bound of the relative entropy of entanglement, which is tight for the already solved examples in two qubits. This bound also becomes an upper bound of the distillable entanglement, since it has been shown that the relative entropy of entanglement is an upper bound of the distillable entanglement [3, 6, 7].

For a given entangled state ϱ, its relative entropy of entanglement [2, 3] is defined as

$$E_R(\varrho) = \min_{\sigma\in\mathcal{D}} S(\varrho\|\sigma) = \min_{\sigma\in\mathcal{D}}[\mathrm{Tr}\,\varrho\log\varrho - \mathrm{Tr}\,\varrho\log\sigma] \tag{1}$$

where the minimization is performed over all density matrices in the set of disentangled states $\mathcal{D}$. The state σ in the set of $\mathcal{D}$ can be written as a convex sum of the product states, and hence

$$\sigma = \sum_i p_i|i_A\rangle\langle i_A|\otimes|i_B\rangle\langle i_B|\otimes|i_C\rangle\langle i_C|\otimes\ldots \tag{2}$$

with $p_i \geqslant 0$ and $\sum_i p_i = 1$. Let us assume that σ^* is the closest disentangled state which minimizes $S(\varrho\|\sigma)$, and hence

$$S(\varrho\|\sigma) \geqslant S(\varrho\|\sigma^*) \tag{3}$$

for any $\sigma \in \mathcal{D}$. Among those disentangled states, we consider the state σ' which is obtained from σ^* by local filtering operations. It is obvious from equation (2) that σ' is also disentangled.

It should be noted that, in the definition of the relative entropy of entanglement, the set of $\mathcal{D}$ is sometimes taken for the positive partial transposed (PPT) states [6], and the state σ^* achieving the minimum should be called the closest PPT state. Even in this case, σ' obtained from σ^* by local filtering is also PPT, since the PPT property is invariant under local filtering operations. Therefore, all the results for the closest disentangled states shown below also hold for the closest PPT states.

Hereafter, we first restrict ourselves to the case of two qubits in order to simplify the discussion. Let us consider Bob's local filtering operation as follows:

$$\sigma' = \frac{(I\otimes \mathrm{e}^{t\vec{n}\cdot\vec{\sigma}/2})\sigma^*(I\otimes \mathrm{e}^{t\vec{n}\cdot\vec{\sigma}/2})}{\mathrm{Tr}\,[(I\otimes \mathrm{e}^{t\vec{n}\cdot\vec{\sigma}/2})\sigma^*(I\otimes \mathrm{e}^{t\vec{n}\cdot\vec{\sigma}/2})]} \tag{4}$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the vector of Pauli matrices, $|\vec{n}| = 1$ (not required though) and t is any real parameter. Using $\log A = \int_0^\infty \frac{xA-1}{A+x}\frac{\mathrm{d}x}{1+x^2}$, the polynomial expansion of $\log(\mathrm{e}^{tB}A\,\mathrm{e}^{tB})$ with respect to t is given by

$$\log(\mathrm{e}^{tB}A\,\mathrm{e}^{tB}) = \log A + t\int_0^\infty \frac{1}{A+x}\{A,B\}\frac{1}{A+x}\,\mathrm{d}x + \mathcal{O}(t^2) \tag{5}$$

where $\{A, B\} \equiv AB + BA$, and therefore

$$\mathrm{Tr}\,\varrho \log \sigma' = \mathrm{Tr}\,\varrho \log \sigma^* + t \left[\mathrm{Tr}\,\varrho \int_0^\infty \frac{1}{\sigma^* + x} \frac{\{\sigma^*, (I \otimes \vec{n} \cdot \vec{\sigma})\}}{2} \frac{1}{\sigma^* + x} \,\mathrm{d}x \right. \\ \left. - \mathrm{Tr}\,[(I \otimes \vec{n} \cdot \vec{\sigma})\sigma^*] \right] + \mathcal{O}(t^2). \tag{6}$$

If the linear coefficient of t is not zero, there always exists σ' satisfying $S(\varrho\|\sigma') < S(\varrho\|\sigma^*)$ for a small enough $|t|$ (σ' is obviously non-singular at $t = 0$), but this contradicts equation (3). Therefore the linear coefficient must be zero for any direction of $\vec{n}$. When Bob's reduction of σ^* is written as

$$\sigma_B^* = \mathrm{Tr}_A \sigma^* = \tfrac{1}{2}[I + \vec{s}_B \cdot \vec{\sigma}] \tag{7}$$

then σ^* must satisfy

$$\vec{n} \cdot \vec{s}_B = \mathrm{Tr}\,\varrho \int_0^\infty \frac{1}{\sigma^* + x} \frac{\{\sigma^*, (I \otimes \vec{n} \cdot \vec{\sigma})\}}{2} \frac{1}{\sigma^* + x} \,\mathrm{d}x. \tag{8}$$

Let $|i\rangle$ be eigenstates of σ^*, and $\sigma^* = \sum_i \lambda_i |i\rangle\langle i|$. Then

$$\begin{aligned} \vec{n} \cdot \vec{s}_B &= \sum_{i,j} \int_0^\infty \frac{\frac{\lambda_i + \lambda_j}{2}}{(\lambda_i + x)(\lambda_j + x)} \,\mathrm{d}x \,\langle i|(I \otimes \vec{n} \cdot \vec{\sigma})|j\rangle\langle j|\varrho|i\rangle \\ &= \sum_{i,j} \langle i|(I \otimes \vec{n} \cdot \vec{\sigma})|j\rangle\langle j|\varrho|i\rangle + \sum_{i,j} \langle i|(I \otimes \vec{n} \cdot \vec{\sigma})|j\rangle\langle j|\varrho|i\rangle g_{ij} \\ &= \vec{n} \cdot \vec{r}_B + \mathrm{Tr}\,(I \otimes \vec{n} \cdot \vec{\sigma})\varrho \circ g. \end{aligned} \tag{9}$$

Here, $\vec{r}_B$ is the Bloch vector of Bob's reduction of ϱ:

$$\varrho_B = \mathrm{Tr}_A \varrho = \tfrac{1}{2}[I + \vec{r}_B \cdot \vec{\sigma}] \tag{10}$$

the matrix g is given by

$$g_{ij} = \begin{cases} \dfrac{\lambda_i + \lambda_j}{2} \dfrac{\log \lambda_i - \log \lambda_j}{\lambda_i - \lambda_j} - 1 & \text{for} \quad \lambda_i \neq \lambda_j \\ 0 & \text{for} \quad \lambda_i = \lambda_j \end{cases} \tag{11}$$

and $A \circ B$ is the Hadamard product defined as

$$[A \circ B]_{ij} = A_{ij} B_{ij}. \tag{12}$$

Since g is real symmetric, $\varrho \circ g$ is Hermitian and the reduction of $\varrho \circ g$ can be written as

$$(\varrho \circ g)_B = \mathrm{Tr}_A(\varrho \circ g) = \tfrac{1}{2} \vec{g}_B \cdot \vec{\sigma}, \tag{13}$$

where $\vec{g}_B$ is a real vector and $\mathrm{Tr}\,(\varrho \circ g) = 0$ was taken into account. Then, since equation (9) must hold for any $\vec{n}$ the reduction of σ^* must satisfy

$$\vec{s}_B = \vec{r}_B + \vec{g}_B. \tag{14}$$

In this way, it can be seen that the local property of the closest disentangled state is strongly related to the extremal condition with respect to the local filtering.

It should be noted here that, since σ^* minimizes $S(\varrho\|\sigma)$, σ^* lies on the boundary between the set of disentangled states and entangled states [8, 9]. In the case of two qubits, the change of the concurrence [10, 11] due to the local filtering has been obtained in [12–14]. According to theorem 1 in [14], if the operator describing the local filtering is full rank (that is our case for any finite t), the state obtained by local filtering from the boundary state also lies on the boundary. Therefore, when t is varied, σ' moves on the boundary surface. Whether the same property holds in any system or not is still an open question, but the crucial fact we have used in this paper is that σ' is always disentangled (and PPT) for any t. That is obviously kept in any system.

Therefore, the above discussion can be extended to any system in a very straightforward manner. For the multi-party system, the local filtering of the type $I \otimes \ldots \otimes e^{t\vec{n}\cdot\vec{\sigma}/2} \otimes \ldots \otimes I$ can be applied to obtain the same result. For the party with d-dimension, the set of Pauli matrices is replaced with the set of $d^2 - 1$ Hermitian generators $\vec{J}$ of $SU(d)$ [15], and we can obtain the condition which the $d^2 - 1$ dimensional generalized Bloch vector of the closest disentangled state must satisfy. Then the following theorem is proved.

Theorem. *Let ϱ be an entangled state in any multi-party system with any dimension. The reduction of the closest disentangled (and PPT) state σ^* with respect to the party X must satisfy $\vec{s}_X = \vec{r}_X + \vec{g}_X$, where $\vec{s}_X$ and $\vec{r}_X$ are the generalized Bloch vectors of σ^*_X and ϱ_X, respectively, and $(\varrho \circ g)_X = \frac{1}{2}\vec{g}_X \cdot \vec{J}$.*

It has been proved in [16], if $E_R(\varrho) = \max\{S(\varrho_A) - S(\varrho), S(\varrho_B) - S(\varrho)\}, \sigma^*$ must have the same reduction as ϱ. According to the above theorem, the condition for which the reductions are the same is given by the following corollary:

Corollary 1. *The closest disentangled (and PPT) state σ^* has the same reduction as ϱ with respect to the party X ($\sigma^*_X = \varrho_X$), if and only if $(\varrho \circ g)_X = 0$.*

Further, if σ^* commutes with ϱ, σ^* is diagonalized in the same basis as ϱ. Since all the diagonal elements of g in this basis are always zero, $\varrho \circ g = 0$ in this case, and hence $(\varrho \circ g)_X = 0$ for every party X. Then the following corollary is proved.

Corollary 2. *Let ϱ be an entangled state in any multi-party system with any dimension. The closest disentangled (and PPT) state σ^* must have the same reduction as ϱ with respect to every party, if σ^* commutes with ϱ.*

Now it is worth checking how the condition of the above theorem ($\vec{s}_X = \vec{r}_X + \vec{g}_X$) is satisfied in analytically solved examples of the relative entropy of entanglement. In all of the already solved examples, it can be seen that $\vec{g}_X = 0$ and the reductions are the same as shown below. Does σ^* commute with ϱ in all examples? The answer is no. In fact, for the pure entangled state in two qubits

$$|\psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle \tag{15}$$

the closest disentangled state is [3]

$$\sigma^* = p|00\rangle\langle 00| + (1-p)|11\rangle\langle 11| \tag{16}$$

which does not commute with $\varrho = |\psi\rangle\langle\psi|$. Instead, we found that all examples satisfy a condition weaker than $[\varrho, \sigma^*] = 0$, that is

$$(|j\rangle\langle j|[\varrho, \sigma^*]|i\rangle\langle i|)_A = (|j\rangle\langle j|[\varrho, \sigma^*]|i\rangle\langle i|)_B = 0 \tag{17}$$

for any i and j. Here, $[A, B] \equiv AB - BA$, and $|i\rangle$ are the eigenstates of σ^*. This condition is also sufficient for $(\varrho \circ g)_A = (\varrho \circ g)_B = 0$, since equation (17) is equivalent to

$$\lambda_i = \lambda_j \qquad \text{or} \qquad (|j\rangle\langle i|)_A \varrho_{ji} = (|j\rangle\langle i|)_B \varrho_{ji} = 0 \tag{18}$$

and hence

$$(\varrho \circ g)_A = \sum_{ij} (|j\rangle\langle i|)_A \varrho_{ji} g_{ji} = 0. \tag{19}$$

Further, depending on how to satisfy the condition, the examples are mainly classified into the following two categories:

(i) $[\varrho, \sigma^*] = 0$ and equation (17) is satisfied (corresponding to corollary 1). The Bell diagonal states in two qubits [2], the maximally entangled mixed states in two qubits [3, 17] and the isotropic state with any dimension [6] belong to this category.

(ii) In the support space of ϱ, $(|j\rangle\langle i|)_A = (|j\rangle\langle i|)_B = 0$ for all $i \neq j$, and equation (17) is satisfied. The maximally correlated states (including pure states) [6, 18] and the state proposed in [19] belong to this category.

It is interesting to note that, if we wash out the classical correlations as well as the quantum correlations, the closest 'uncorrelated' state is $\sigma_u = \varrho_A \otimes \varrho_B \otimes \varrho_C \cdots$ [2], where the reductions of σ_u are always the same as ϱ. In the case of the closest disentangled state, although there is no guarantee that the reductions are the same, $(\varrho \circ g)_X = 0$ is rather widely satisfied and reductions are the same in many cases as is shown above. This fact might originate from the properties of the relative entropy. In fact, if we adopt the Bures metric

$$B(\varrho\|\sigma) = 2 - 2\mathrm{Tr}\sqrt{\sigma}\varrho\sqrt{\sigma} \tag{20}$$

as the distant measure, using $\sqrt{A} = \frac{1}{\pi}\int_0^\infty \frac{A}{A+x}\frac{\mathrm{d}x}{\sqrt{x}}$, we obtain

$$(\vec{n}\cdot\vec{s}_B)(\mathrm{Tr}\sqrt{\sigma^*}\varrho\sqrt{\sigma^*}) = \mathrm{Tr}\int_0^\infty \frac{1}{\sigma^*+x}\frac{\{\sigma^*, (I\otimes\vec{n}\cdot\vec{\sigma})\}}{2}\frac{1}{\sigma^*+x}\{\varrho, \sqrt{\sigma^*}\}\frac{\sqrt{x}}{\pi}\,\mathrm{d}x. \tag{21}$$

From the above, it seems to be unlikely that $\vec{s}_B = \vec{r}_B$ in many cases.

Let us return to the problem minimizing the relative entropy. Instead of the local filtering, we can consider the local unitary transformation as follows:

$$\sigma' = (I\otimes \mathrm{e}^{\mathrm{i}t\vec{n}\cdot\vec{\sigma}/2})\sigma^*(I\otimes \mathrm{e}^{-\mathrm{i}t\vec{n}\cdot\vec{\sigma}/2}) \tag{22}$$

which also ensures that σ' is disentangled (and PPT) for any t. Expanding the right-hand side of the above equation with respect to t, and the same discussion as in the local filtering case gives

$$\begin{aligned}\mathrm{Tr}\,\varrho\int_0^\infty &\frac{1}{\sigma^*+x}\frac{[\sigma^*, (I\otimes\vec{n}\cdot\vec{\sigma})]}{2}\frac{1}{\sigma^*+x}\,\mathrm{d}x\\ &= \frac{1}{2}\sum_{ij}\langle i|(I\otimes\vec{n}\cdot\vec{\sigma})|j\rangle\langle j|\varrho|i\rangle(\log\lambda_j - \log\lambda_i) = 0\end{aligned} \tag{23}$$

and hence

$$([\varrho, \log\sigma^*])_B = \frac{\mathrm{i}}{2}\vec{h}_B\cdot\vec{\sigma} = 0 \tag{24}$$

with $\vec{h}_B$ a real vector.

Therefore, the closest disentangled (and PPT) state must satisfy both equations (14) and (24) and Alice's counterparts. It is interesting to note that, even though $S(\varrho\|\sigma^*) \neq S(\varrho\|\sigma^*_{PPT})$ where σ^* and σ^*_{PPT} is the closest disentangled and PPT state of ϱ, respectively, both σ^* and σ^*_{PPT} satisfy the same equations of (14) and (24) (and Alice's counterparts). The total number of these equations in the $d\otimes d$ bipartite system is $4(d^2-1)$. Therefore, *in principal*, d^4-1 independent parameters in σ^* can be reduced to d^4-4d^2+3 by solving those equations. In the case of the simplest $2\otimes 2$ systems, the number of remaining parameters is only three. Unfortunately, however, both equations (14) and (24) are not still tractable. In order to determine g, for example, the eigenvectors $|i\rangle$ and eigenvalues λ_i of σ^* are needed, although the purpose is to search for σ^*.

However, one of the important facts about the relative entropy of entanglement is that $E_R(\varrho)$ gives an upper bound of the distillable entanglement of ϱ [3, 6, 7]. Since the analytical calculation of $E_R(\varrho)$ is a hard problem, it might also be worthwhile to suggest a prescription for obtaining an upper bound of $E_R(\varrho)$, which is also an upper bound of the distillable entanglement. For this purpose, we induce some constraints to the minimization problem of $S(\varrho\|\sigma)$. The constraints we induce are

$$\vec{s}_A = \vec{r}_A \qquad \vec{s}_B = \vec{r}_B \tag{25}$$

and

$$([\varrho, \sigma^*])_A = ([\varrho, \sigma^*])_B = 0. \tag{26}$$

The advantage of adopting equations (25) and (26) is that, since these relations are satisfied in all of the analytically solved examples as shown before (equation (26) was obtained by summing i and j in (17), and thus (26) is weaker than (17)), the obtained upper bound exactly agrees with $E_R(\varrho)$ for those states. Therefore, this upper bound is expected to be good for the other states. Further, if the obtained σ^* happens to satisfy equation (17), the extremal conditions of both local filtering and local unitary operation are ensured, although these extremal conditions are not generally satisfied in this approximate method.

Let us restrict ourselves to the $2 \otimes 2$ systems, and let a Hilbert–Schmidt representation of ϱ and σ^* be

$$\begin{aligned} \varrho &= \frac{1}{4}\left(I \otimes I + \vec{r}_A \cdot \vec{\sigma} \otimes I + I \otimes \vec{r}_B \cdot \vec{\sigma} + \sum_n \hat{t}_{nn}\sigma_n \otimes \sigma_n\right) \\ \sigma^* &= \frac{1}{4}\left(I \otimes I + \vec{r}_A \cdot \vec{\sigma} \otimes I + I \otimes \vec{r}_B \cdot \vec{\sigma} + \sum_{n,m} \hat{\tau}_{nm}\sigma_n \otimes \sigma_m\right) \end{aligned} \tag{27}$$

where ϱ was chosen to be a canonical form (T-matrix $\hat{t}$ is diagonalized by a suitable local unitary transformation [20]) and we adopted equations (25). Then, simple calculations show that equation (26) is equivalent to

$$\begin{cases} \hat{t}_{ii}\hat{\tau}_{ij} - \hat{t}_{jj}\hat{\tau}_{ji} = 0 \\ \hat{\tau}_{ij}\hat{t}_{jj} - \hat{\tau}_{ji}\hat{t}_{ii} = 0. \end{cases} \tag{28}$$

This implies that $\hat{\tau}_{ij} = \hat{\tau}_{ji}$ for $\hat{t}_{ii} = \hat{t}_{jj}$ and $\hat{\tau}_{ij} = 0$ for $\hat{t}_{ii} \neq \hat{t}_{jj}$. Therefore, $\hat{\tau}$ must be real symmetric and if t_{ii} are not degenerate at all, all the off-diagonal elements of $\hat{\tau}$ must vanish. Further, since the off-diagonal element (say $\hat{\tau}_{xy}$) is non-vanishing only when $\hat{t}_{xx} = \hat{t}_{yy}$, a suitable local unitary transformation simultaneously applied to σ^* and ϱ, which rotates the x–y space of the T-matrix, makes it possible to simultaneously diagonalize $\hat{t}$ and $\hat{\tau}$ (the state $(U_A \otimes U_B)\sigma^*(U_A^\dagger \otimes U_B^\dagger)$ is minimum for $(U_A \otimes U_B)\varrho(U_A^\dagger \otimes U_B^\dagger)$ by the property of the relative entropy). This implies that σ^* of all the analytically solved examples in two qubits shown before can be written in a canonical Hilbert–Schmidt form, when ϱ is chosen to be a canonical form by selecting a suitable local unitary transformation.

Since the Bloch vector of each reduction of σ^* is the same as ϱ, the number of undetermined parameters is three: $\hat{\tau}_{11}$, $\hat{\tau}_{22}$ and $\hat{\tau}_{33}$. Obviously, we have not explicitly used the condition that σ^* must be disentangled, yet. According to proposition 2 in [20], $\vec{\tau} = (\hat{\tau}_{11}, \hat{\tau}_{22}, \hat{\tau}_{33})$ must belong to Horodecki's octahedron $\mathcal{L}$. Although this separability condition is sufficient for $\vec{r}_A = \vec{r}_B = 0$ [20], the geometry of the boundary in T-space is not simple in general [21, 22]. Therefore, although the difficulty of the complicated structure of the entangled–disentangled boundary is not still avoided even in this approximate method, a reasonably good upper bound of $E_R(\varrho)$ can be obtained by minimizing only three parameters in T-space.

It should be noted briefly about the possibility of extension of this approximate method to higher dimensional systems. The total number of equations in equations (25) and (26) in the

$d \otimes d$ bipartite system is $4(d^2 - 1)$, which is the same as the number of extremal conditions of local filtering and local unitary. As a result, $d^4 - 4d^2 + 3$ parameters remain undetermined. Since the dimension of the T-matrix in the $d \otimes d$ system is $d^2 - 1$, some off-diagonal elements as well as the diagonal elements in T-matrix necessarily remain undetermined for $d \geqslant 3$.

To conclude, we study the extremal condition with respect to local filtering. We obtained the set of equations which both the closest disentangled and PPT state must satisfy, and showed that the local property of the closest disentangled (and PPT) state is strongly related to the extremal condition of the local filtering. Further, we obtained the sufficient condition for which the closest disentangled state has the same reduction as the given entangled state, and showed that the condition has been rather widely satisfied. Further, in the case of two qubits, we suggest a prescription to obtain an upper bound of the relative entropy of entanglement, which is tight for the analytically already solved examples in two qubits.

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